Group actions on Banach spaces and a geometric characterization of a-T-menability

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Abstract

We prove a geometric characterization of a-T-menability through proper, affine, isometric actions on the Banach spaces $L_p[0,1]$ for $1 < p < 2$. This answers a question of A. Valette.

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Let X be a normed space. An *affine, isometric action* of a group Γ on X is defined as $\Psi(g)v = \pi(g)v + \gamma(g)$ for $v \in X$, $g \in \Gamma$, where π is a unitary (i.e. linear isometric) representation of Γ on X and $\gamma : \Gamma \to X$ satisfies the cocycle identity with respect to π, i.e. $\gamma(gh) = \pi(g)\gamma(h) + \gamma(g)$. The action is *proper* if $\lim_{g\to\infty} \|\Psi(g)v\| = \infty$ for every $v \in X$. This is equivalent to $\lim_{g\to\infty} \|\gamma(g)\| = \infty$. One can express this idea in the language of coarse geometry by saying that every orbit map is a coarse embedding.

The following definition is due to Gromov.

Definition 1 ([Gr, 6.A.III]) *A second countable, locally compact group is said to be* a-T-menable *(has the* Haagerup approximation property*) if it admits a proper, affine, isometric action on a separable Hilbert space $\mathcal{H}$.*

A-T-menability was designed as a strong opposite of Kazhdan's property (T). We recall here a geometric characterization of property (T) known as the Delorme-Guichardet Theorem, for a detailed account of the subject see [BHV].

Definition 2 *A second countable, locally compact group Γ has* Kazhdan's Property (T) *if and only if every affine isometric action of Γ on a Hilbert space has a fixed point.*

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As suggested in the definition, a-T-menability turned out to be equivalent to the Haagerup property (this was proved in [BCV]), which arose in the study of approximation properties of operator algebras and has application to harmonic analysis. There are many other characterizations of a-T-menability, in particular Gromov showed [Gr, 7.A] that it is equivalent to existence of a proper isometric action on the (either real or complex) infinite dimensional hyperbolic space.

Recently N. Brown and E. Guentner [BG] proved that every discrete group admits a proper, affine and isometric action on an ℓ_2-direct sum $(\sum \ell_{p_n})_2$, for some sequence $\{p_n\}$ satisfying $p_n \longrightarrow \infty$. Since there are discrete groups which are not a-T-menable, i.e. groups which are Kazhdan (T), an existence of a proper, affine, isometric action on a reflexive Banach space does not in general imply a-T-menability. Also results of G. Yu show that property (T) groups may admit proper, affine, isometric actions on the spaces ℓ_p for $p > 2$ [Yu]. We also refer the reader to the recent article [BFGM] for a thorough study of similar questions in the context of property (T).

What we are interested in is to find Banach spaces actions on which imply or characterize a-T-menability. The motivation comes from a question of A.Valette, who in [CCJJV, Section 7.4.2] asked whether there are geometric characterizations of a-T-menability other than through actions on infinite-dimensional hyperbolic spaces. We prove the following

Theorem 3 *For a second countable, locally compact group* Γ *the following conditions are equivalent:*

(1) Γ *is a-T-menable*

(2) Γ *admits a proper, affine, isometric action on the Banach space* $L_p[0,1]$ *for some* $1 < p < 2$

(3) Γ *admits a proper, affine, isometric action on the Banach space* $L_p[0,1]$ *for all* $1 < p < 2$

Note that the results in [BG,Yu] show that Theorem 3 cannot be extended to $p > 2$ or to the class of reflexive or uniformly convex Banach spaces.

We also want to mention a problem raised in [Gr, 6.D$_3$] by Gromov: for a given group Γ find all such $p \geq 1$ for which Γ admits a proper, affine, isometric action on ℓ_p. Our methods give some partial information on possible answers to this question, namely Proposition 8 states that only a-T-menable groups may admit such actions on ℓ_p for $0 < p < 2$.

A-T-menability is an important property in studying the Baum-Connes Conjecture. N. Higson and G. Kasparov showed [HK] that every discrete a-T-

menable group satisfies the Baum-Connes Conjecture with arbitrary coefficients.

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1 Proofs

We will use the fact that a-T-menability can be characterized in terms of existence of certain conditionally negative definite functions, which we define now.

By a kernel on a set X we mean a symmetric function $K : X \times X \to \mathbb{R}$.

Definition 4 *A kernel K is said to be* conditionally negative definite *if*

$$\sum K(x_i, x_j) c_i c_j \leq 0$$

for all $n \in \mathbb{N}$ and $x_1, ..., x_n \in X$, $c_1, ..., c_n \in \mathbb{R}$ such that $\sum c_i = 0$.

A function $\psi\colon \Gamma \to \mathbb{R}$ on a metric group Γ, satisfying $\psi(g) = \psi(g^{-1})$ is said to be conditionally negative definite if $K(g,h) = \psi(gh^{-1})$ is a conditionally negative definite kernel.

It is easy to check that if $(\mathcal{H}, \|\cdot\|)$ is a Hilbert space then the kernel $K(x,y) = \|x-y\|^2$ is conditionally negative definite.

The following characterization is due to M.E.B. Bekka, P.-A. Cherix and A. Valette.

Theorem 5 ([BCV]) *A second countable, locally compact group Γ is a-T-menable if and only if there exists a continuous, conditionally negative definite function $\psi : \Gamma \to \mathbb{R}_+$ satisfying $\lim_{g\to\infty} \psi(g) = \infty$.*

To prove Theorem 3 we also need the following lemmas concerning conditionally negative definite functions and kernels on L_p-spaces. These facts where proved by Schoenberg [Sch], for a further discussion see e.g. [BL, Chapter 8].

Lemma 6 *Let K be a conditionally negative definite kernel on X and $K(x,y) \geq 0$ for all $x,y \in X$. Then the kernel K^{α} is conditionally negative definite for any $0 < \alpha < 1$.*

Proof. Let K be a conditionally negative definite kernel. Then for every $t \geq 0$ the kernel $1 - e^{-tK} \geq 0$ is also conditionally negative definite and we have

$$\int_0^{\infty} \left(1 - e^{-tK}\right) d\mu(t) \geq 0$$

for every positive measure μ on $[0, \infty)$. For every $x > 0$ and $0 < \alpha < 1$ the following formula holds

$$x^{\alpha} = c_{\alpha} \int_0^{\infty} \left(1 - e^{-tx}\right) t^{-\alpha-1} dt,$$

where c_{α} is some positive constant. Thus K^{α} is also a conditionally negative definite kernel for every $0 < \alpha < 1$. $\square$

Lemma 7 *The function $\|x\|^p$ is conditionally negative definite on $L_p(\mu)$ when $0 < p \leq 2$.*

Proof. The kernel $|x - y|^2$ is conditionally negative definite on the real line (as a square of the metric on a Hilbert space). By Lemma 6, for any $0 < p \leq 2$ the kernel $|x - y|^p$ is also conditionally negative definite on $\mathbb{R}$, i.e.,

$$\sum |x_i - x_j|^p c_i c_j \leq 0$$

for every such p, all $x_1, ..., x_n \in \mathbb{R}$ and $c_1, ..., c_n \in \mathbb{R}$ such that $\sum c_i = 0$. Integrate the above inequality with respect to the measure μ to establish the proof. $\square$

It follows from the lemmas that the norm on $L_p(\mu)$ is a conditionally negative definite function, provided $1 \leq p \leq 2$.

To state the next proposition we define a more general notion of a proper action, it is necessary when talking about the spaces $L_p(\mu)$ for $p < 1$ which are not normable metric vector spaces. Thus, if X is just a metric space we call an isometric action of Γ on X proper if the set $\{\, g \in \Gamma \,|g\mathcal{U} \cap \mathcal{U} \,\}$ is finite for any bounded set $\mathcal{U} \subset X$. For normed spaces this is consistent with the definitions stated in the introduction.

Proposition 8 *If a second countable, locally compact group* Γ *admits a proper, affine, isometric action on a space* $L_p(\mu)$ *for some* $0 < p < 2$ *then* Γ *is a-T-menable.*

Proof. Given a proper, affine, isometric Γ-action on $L_p[0,1]$ consider the function $\psi : \Gamma \to \mathbb{R}$, $\psi(g) = \|\gamma(g)\|^p$, where γ is the cocycle associated with the action. Since the p-th power of the norm on $L_p[0,1]$ is a conditionally negative definite function by Lemma 7, ψ is a conditionally negative function on Γ. The considered Γ-action is proper thus $\lim_{g\to\infty} \|\gamma(g)\|^p = \infty$ and by Theorem 5, Γ is a-T-menable. $\square$

In particular only a-T-menable groups may admit proper, affine isometric actions on the spaces ℓ_p for $0 < p < 2$ (cf. [Yu]).

Proof of Theorem 3. (1) $\Rightarrow$ (3). Let G be a locally compact, second countable, a-T-menable group. Then by [CCJJV, Theorem 2.2.2] there exists a standard probability space (X, μ) and a measure preserving action of G on X such that

(1) there exists a sequence of Borel sets $A_n \subseteq X$ such that $\mu(A_n) = \frac{1}{2}$ and $\sup_{g\in B(e,n)} \mu(A_n g \triangle A_n) \leq \frac{1}{2^n}$,
(2) the action is strongly mixing, i.e. $\langle f, f \cdot g\rangle \to 0$ when $g \to \infty$ for every $f \in L_2(X,\mu)$ such that $\int f d\mu = 0$.

Choose the sequence Let $v_n(x) = 1_{A_n}(x) - \frac{1}{2} \in L_2(X,\mu)$. Then $\|v_n\|_2 = \dfrac{1}{2}$ and

$$\int_X v_n(x)\ d\mu = 0$$

so by strong mixing,

$$\|v_n - v_n \cdot g\|_2 \to \sqrt{2}\|v_n\|_2,$$

when $g \to \infty$. Also, for $g \in B(e,n)$ we have

$$\|v_n - v_n \cdot g\|_2 = \mu(A_n g \triangle A_n) \leq \frac{1}{2^n}$$

for all $g \in B(e,n)$.

Now given $p < 2$ define

$$w_n(x) = |v_n(x)|^{2/p} \operatorname{sign}(v_n(x)) \in L_p(X,\mu).$$

In other words, w_n is a image of v_n under the *Mazur map*, which is a uniform homeomorphism between unit balls of L_p-spaces, see [BL, Ch. 9.1] for details

and estimates. Moreover this map clearly commutes with the regular representation. By the uniform continuity of the Mazur map and its inverse there exist constants $C, \delta > 0$ (which depend only on p) such that the sequence w_n satisfies

(1) $\sup_{g\in B(e,n)} \|w_n \cdot s - w_n\|_p \leq C/2^n$,
(2) $\|w_n \cdot g - w_n\|_p \geq \delta$ for all $g \in G \setminus B(e, S_n)$ for some $S_n > 0$, which depends on n only (the sequence $\{S_n\}$ can be chosen to be increasing).

This allows to construct a proper affine isometric action on $L_p(X,\mu)$ in a standard way. Define $b : G \to \left(\bigoplus_{n=1}^{\infty} L_p(X,\mu)\right)_p$ (p denotes the L_p-norm on the infinite direct sum)

$$b(g) = \oplus_{n=1}^{\infty}\ \rho(g)w_n - w_n$$

where $\rho : G \to \mathrm{Iso}(L_p(X,\mu))$ is the right regular representation of G on X. Then b is a cocycle for the representation $\oplus\rho$ by standard calculations (see e.g. [BCV]).

This way we obtain a proper isometric action on $\left(\bigoplus_{n=1}^{\infty} L_p(X,\mu)\right)_p$ and the only thing left to notice is that by construction in the proof of [CCJJV, Theorem 2.2.2] the measure μ is non-atomic, thus by the isometric classification of L_p-spaces, $L_p(X,\mu)$ is isometric to $L_p[0,1]$ and the p-sum of infinitely many of these spaces is again isometric to $L_p[0,1]$. Thus G admits a proper, affine, isometric action on $L_p[0,1]$.

(3) $\Rightarrow$ (2). This is obvious.

(2) $\Rightarrow$ (1). This implication is proved in Proposition 8 above. □

Note that the above methods cannot be applied to other Banach spaces. J. Bretagnolle, D. Dacuhna-Castelle and J.L. Krivine showed [BDCK] that the function $\|x\|^p$, $0 < p \leq 2$, is a conditionally negative definite kernel on a Banach space X if and only if X is isometric to a subspace of $L_p(\mu)$ for some measure μ. Together with Lemma 6 this covers all powers $p \geq 1$.

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